



Adaptive Expectation for Network Goods

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Abstract

With regard to the network goods, the customer's utility is affected by the network size. However, from asymmetric information, customers speculate the network size through their expectations. From literature, the rational and static expectations are commonly studied. This paper employs a well-known expectation type, called adaptive expectation, into a network good in the discrete time framework. Moreover, several properties of equilibrium point are investigated such as location, stability, oscillation, and initial point. Key result is that the properties of equilibrium point depend on the type of expectation.

Keywords: Network externality, Adaptive expectation, Fixed point theorem, Stability JEL Classification: D01, D11, D84, D85

1. Introduction

With regard to the network goods, the customer's utility is affected by the network size. The network externality concept is successfully utilized in the study of economics of network industries (Shy, 2001). We remark that the concept of network externality was introduced by Bell's employee, N.Lytkins (1917). The network externality function is the function that describes the relationship between network value and its corresponding size. However, from asymmetric information, customer speculates the network size through his expectation. To locate the equilibrium points; the rational expectation is generally employed, (Amir and Lazzati, 2011), (Easley and Kleinberg, 2010), (Katz and Shapiro, 1985). To examine the dynamic approach, the simplest expectation, called static expectation, is generally employed, (Easley and Kleinberg, 2010), (Grajek, 2008). In their conclusions, the location and stability condition of equilibrium points are regardless of the type of expectation. We review both expectations in Section II. In Section III, we employ the other well-known expectation, called adaptive expectation; and introduce a new expectation type by modifying the traditional adaptive expectation; and compare the location of equilibrium points for four expectations. In Section VI, we investigate and compare the stability and oscillation condition of equilibrium point for four expectations. The key result is that under a certain condition, the stability is also dependent on the type of expectation. With regard to the discrete time framework, we have opportunity to discuss the stability and oscillation region of stable equilibrium points of difference equation in Section V.

2. Model

We analyze an underlying network good model in discrete time, $t \in \{0,1,2, \dots\}$, and all variables that is subscribed t are time variant, otherwise time invariant. For all t , each customer's utility is composed of his individual preference and the network externality value which depends on his expectation of network size. In this paper, for simplicity, we treat the network size as the market share; that is, it has value between zero and one, $N_t \in [0,1]$. Hence,

$$u(v, N_t^e) = v + g(N_t^e) \tag{2.1}$$

where v is the individual preference, N_t^e is the expectation of network size at time t , and g is network externality function. We further make three assumptions.

1. The individual preference v is distributed according to a cumulative distribution function (cdf), F , and a probability density function (pdf), f .
2. Each customer has identical expectation and network externality function.
3. The network externality function is an S-shaped function, see Figure 2.1a.

Suppose the price of network good is p ; and the customer will join the network when his utility is greater than the price, $v + g(N_t^e) > p$; otherwise he will stay out. Let v^* be the individual preference level that the customer is indifferent between joining or staying out, $v^* + g(N_t^e) = p$. Thus at time , the equilibrium of network size is

$$N_t = \int_{v^*}^{\infty} f(v)dv = 1 - F(v^*) = 1 - F(p - g(N_t^e)) = h(N_t^e). \tag{2.2}$$

The network size is illustrated by the shaded area under f in Figure 2.1b. Since the first derivative of function h is

$$h'(N_t^e) = f(v^*)g'(N_t^e) > 0; \quad (2.3)$$

and the assumption of network externality function, $g' > 0$, The function, $h: [0,1] \rightarrow [0,1]$, is an increasing

function. Clearly, the location(s) of equilibrium point(s) is(are) determined by the solution of (2.2) and its(their) properties are dependent on function h and the expectation type. The following subsections explain about two common expectation types for network good.

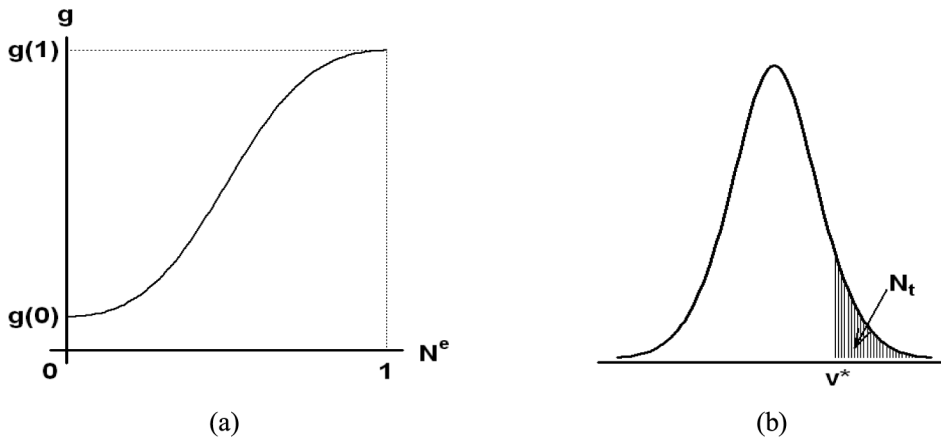


Figure 2.1 (a) is the S-shape network externality function; (b) is the pdf of individual preference of customers and the shading area is the network size at equilibrium.

2.1 Rational Expectation

The rational expectation (RE) is defined as the optimal forecast. All customers use all available information to form their expectation. We remark that RE is originally introduced by John F. Muth (1961). Also, in network good, Katz and Shapiro (1985) employed RE in their paper and called a fulfilled expectation. In this scenario, all customers have perfect foresight about the network size, $N_t^e = N_t$. Consequently, the equilibrium condition (2.2) of the RE scenario is followed by

$$N_t = h(N_t) \quad (2.4)$$

In this case, the equilibrium point(s) is(are) the intersection of function h and 45 degree line. The equation (2.4) is well-known in mathematics and the solution(s) is(are) called as the fixed point(s) of function h . Obviously, the existence of fixed point depends on characteristic of function h .

Proposition 1. If g and F are continuous increasing functions and $v_{min} + g(1) \leq p \leq v_{max} + g(0)$, then h has a fixed point, $\bar{N} = h(\bar{N})$.

Proof. Because h and F are increasing function, then

$$F(v_{min} + g(1) - g(N_t)) \leq F(p - g(N_t)) \leq F(v_{max} + g(0) - g(N_t))$$

$$F(v_{min}) \leq h(N_t) \leq F(v_{max})$$

$$0 \leq h(N_t) \leq 1$$

And since h and g are continuous function, thus h is a real-value continuous function on the interval $[0,1]$. Define $A(N_t) = h(N_t) - N_t$. Then, A is also continuous with $A(0) > 0$ and $A(1) < 0$. Therefore, by Intermediate Value Theorem, there is an $\bar{N} \in (0,1)$ such that $A(\bar{N}) = 0$, hence $\bar{N} = h(\bar{N})$

For economics interpretation, if the price is too low, $p < v_{min} + g(1)$, all customers will join the network which makes $\bar{N} = h(\bar{N}) = 1$. On the other hand, if the price is too high, $p > v_{max} + g(0)$, all customers will stay out which makes $\bar{N} = h(\bar{N}) = 0$. When customers form RE in a network good, the multiple fixed points are commonly discussed. More discussion of these equilibriums can be found in Amir and Lazzati (2011), Easley and Kleinberg (2010) and Katz and Shapiro (1985). In general, there are two disadvantages for RE scenario. First, it is too strong of an assumption that all customers have perfect information about market equilibrium.

Second, because of the instantaneous adjustment, it does not make sense to interpret the adjustment process and analyze in dynamic approach.

Illustration 1: Let us consider a particular example in which $v \sim Uniform(0,3)$ and $g(N_t) = \frac{2N_t^2}{1-2N_t+2N_t^2}$. In this simple case, $h(N_t) = 1 - \frac{p}{3} + \frac{g(N_t)}{3}$. From proposition 2.1, the appropriate price range is $2 < p < 3$. For this concrete example, if $p = 2.5$, there are three equilibrium points at $\bar{N}_L = 0.2113, \bar{N}_M = 0.5000$ and $\bar{N}_H = 0.7887$. Since, $h'(\bar{N}_M) = 1.3333 > 1$ and $h'(\bar{N}_H) = 0.4983 < 1$, \bar{N}_L and \bar{N}_H is lower and higher stable point, respectively, but \bar{N}_M is unstable. If $p = 2.25$ and $p = 2.75$, there is one stable equilibrium point at $\bar{N} = 0.9102$ and $\bar{N} = 0.0897$, respectively. Clearly, the price level affects the number of equilibrium points, when the price increases, the function h moves downward, $\frac{\partial h}{\partial p} = -\frac{1}{3} < 0$. See

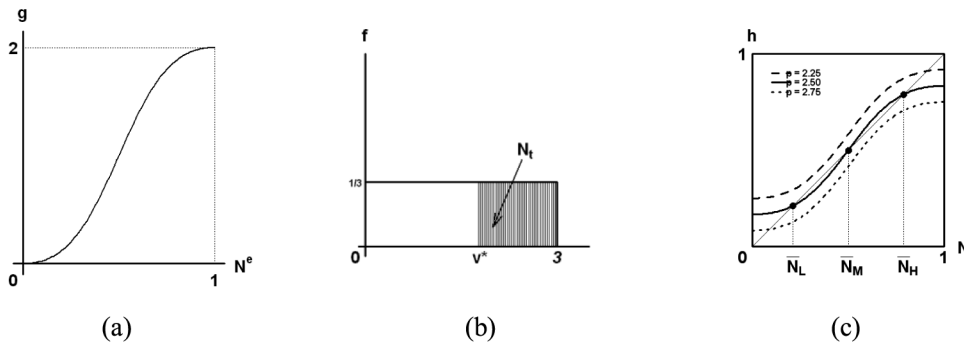


Figure 2.2 (a) is the S-shape network externality function ; (b) is the pdf of individual preference of customers and the shading area is the network size at equilibrium; and (c) is the function h and the equilibrium point(s) at price = 2.25, 2.5 and 2.75.

2.2 Static Expectation

To study the adjustment process in the neighborhood of equilibrium point, the simplest common type of expectation is assumed, called static (or naïve) expectation (SE). All customers use the previous actual value as their present expectation, $N_t^e = N_{t-1}$. They have

asymmetric information in time and expect nothing change for present time. Consequently, the equilibrium condition (2.2) of the SE scenario is developed to the first order nonlinear difference equation,

$$N_{t+1} = h(N_t) \tag{2.5}$$

which a sequence $\{N_t\}_{t=0}^{\infty}$ is a solution and $\lim_{n \rightarrow \infty} N_t = \bar{N}$ is an equilibrium point (steady state) of (2.5). The assumption of a perception lag is both realistic and necessary to derive an empirically tractable strategy to identify the adjustment process to equilibrium point. We can illustrate the stability condition of equilibrium point by pictorial method. Suppose N_0 is the initial network size. Clearly, $N_1 = h(N_0)$, $N_2 = h(N_1)$ and so on. In this method, we track the initial point (N_0, N_0) and move vertically until we reach the curve to $h(N_0) = N_1$, then move horizontally until we reach (N_1, N_1) and so on until we reach the equilibrium point \bar{N} , see Figure 2.3a. According to the illustration 1, the pictorial method tells us that the adjustment process will converge to two

stable equilibrium points which $h'(\bar{N}) < 1$; and diverge from unstable equilibrium point which $h'(\bar{N}) > 1$. Since, $h'(\bar{N}_L) = 0.1546 < 1$, and $h'(\bar{N}_M) = 1.3333 > 1$, \bar{N}_L and \bar{N}_H is lower and higher stable point, respectively, but \bar{N}_M is unstable, see Figure 2.3b. The stability condition will be discussed more in Section IV. If all customers follow this scenario, then, at a steady state, they cannot make any adjustments, which converges precisely to the stable equilibrium point, \bar{N} . Hence, the equation (2.4) is the long run solution of equation (2.5). Moreover, is an increasing function, then h is an increasing function, then there is no chaotic situation for this model (Chiarella, 1988). More discussion of this expectation can be found in Easley and Kleinberg (2010) and Grajek (2008).

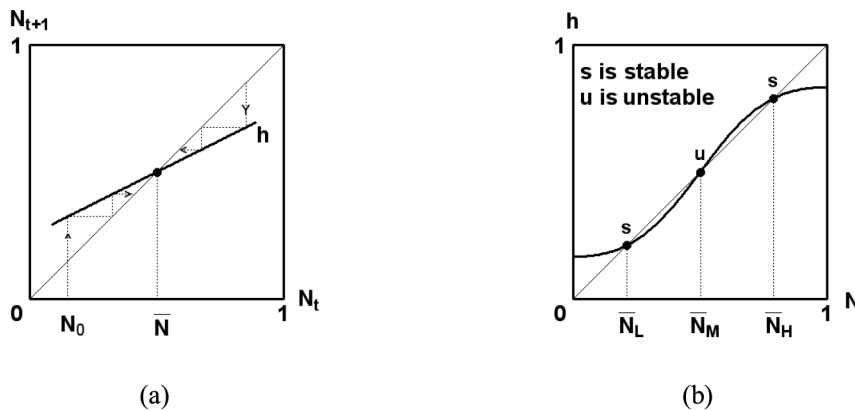


Figure 2.3 (a) is the pictorial method of adjustment process of stable equilibrium point; and (b) shows three equilibrium points, two stable and one unstable.

3. Adaptive Expectation

In this section, we utilize the other well-known expectation, adaptive expectation (AE) to form the expectation of network size. The AE is one of the backward-looking expectations. More precisely, the current expectation of an endogenous variable is directly adjusted adaptively by the weighted mean of its previous actual value and expectation. However, in this paper, we split up AE into two types.

3.1 Adaptive expectation with current information (CAE)

This is the traditional adaptive expectation, that is, all customers expect the future network size by weighting the current expectation and actual network size,

$$N_{t+1}^e = (1 - \omega)N_t^e + \omega N_t \quad (3.1)$$

where $0 < \omega \leq 1$ is the speed of adjustment in which SE is a special case of CAE ($\omega = 1$). Hence, the equilibrium condition (2.2) of CAE scenario is also developed to the first order nonlinear difference equation,

$$N_{t+1}^e = (1 - \omega)N_t^e + \omega h(N_t^e) = H(N_t^e). \quad (3.2)$$

Unlike SE scenario, we analyze the equilibrium condition in terms of expected network size instead of actual network size as in (2.5). The next proposition proves that at a steady state the actual network size is the same as the expected network size.

Proposition 2. Comparison of H and h

- a. H and h have the same equilibrium point.
- b. H is paralleled to h , if h is paralleled to 45 degree line.
- c. If speed of adjustment decreases, H squeezes toward 45 degree line around equilibrium points.

Proof a. From (3.1), $\lim_{t \rightarrow \infty} N_t = \bar{N} \Rightarrow \lim_{t \rightarrow \infty} N_{t+1}^e = \bar{N}$ and from (2.2), $\lim_{t \rightarrow \infty} N_{t+1}^e = \bar{N} \Rightarrow \lim_{t \rightarrow \infty} N_t = \bar{N}$. Hence, $\lim_{t \rightarrow \infty} N_t = \bar{N} \Leftrightarrow \lim_{t \rightarrow \infty} N_{t+1}^e = \bar{N}$. Let \bar{N} be the equilibrium point of h , then, $H(\bar{N}) = (1 - \omega)\bar{N} + \omega h(\bar{N}) = \bar{N}$

Proof b. The first derivative of (3.2) is

$$H' = 1 - \omega + \omega h'. \quad (3.3)$$

Hence, $H' - 1 = \omega(h' - 1)$ and $H' = 1 \Leftrightarrow h' = 1$

Proof c. From (3.3), $\frac{\partial H'}{\partial \omega} = h' - 1$. Hence, when ω decreases, if $h' > 1$, rotates clockwise around equilibrium point, otherwise it rotates counterclockwise around equilibrium point.

Thus, from proposition 2a, we can analyze the properties of neighborhood of equilibrium of actual network size through expected network size instead.

3.2 Adaptive expectation with lagged information (LAE)

We modify an assumption of the CAE. While, in CAE, all customers use the current actual network size to adjust their expectation of the next period, but in LAE, they do not know the current information about network size, but use the previous period one instead. This assumption is more realistic for customers. Thus, the equation (3.1) is modified to

$$N_{t+1}^e = (1 - \omega)N_t^e + \omega N_{t-1} \quad (3.4)$$

where ω is the speed of adjustment as before. Hence, the equilibrium condition (2.2) is developed to the second order nonlinear difference equation,

$$N_{t+1}^e = (1 - \omega)N_t^e + \omega h(N_{t-1}^e) = \mathcal{H}(N_t^e, N_{t-1}^e). \quad (3.5)$$

As $t \rightarrow \infty$, the equilibrium points of LAE are the same as CAE, SE and RE. In other words, the equilibrium points of network size are independent of the type of expectation.

Table 3.1 Summary the meaning and equilibrium condition of four expectations

Type	Meaning	Equilibrium Condition
RE	$N_{t+1}^e = N_{t+1}$	$N_t = h(N_t)$
SE	$N_{t+1}^e = N_t$	$N_{t+1} = h(N_t)$
CAE	$N_{t+1}^e = (1 - \omega)N_t^e + \omega N_t$	$N_{t+1}^e = H(N_t^e)$
LAE	$N_{t+1}^e = (1 - \omega)N_t^e + \omega N_{t-1}$	$N_{t+1}^e = \mathcal{H}(N_t^e, N_{t-1}^e)$

4. Stability

In this section, we study the stability condition of equilibrium points for four expectations. Because of the variety of its definition, let us restate it again here.

1. An equilibrium point, \bar{N} is called Lyapunov stable if for any closed enough initial points N_0 , then N_t stays close to \bar{N} for all time.

2. A Lyapunov stable equilibrium point \bar{N} is called locally asymptotically stable if for any closed enough initial points N_0 , then N_t converges to \bar{N} .

3. A Lyapunov stable equilibrium point is called globally asymptotically stable if for any initial points N_0 , then N_t converges to \bar{N} .

We remark that the term “asymptotic” is generally omitted in economics, while it is required in mathematics. By contrast, the term “Lyapunov” is used in economics but omitted in mathematics. In this paper, our stability of equilibrium point is the same as asymptotic stability, locally and globally. Now we are ready to investigate the stability condition for each expectation type, starting from RE scenario. Though the RE scenario is not dynamic approach, the stability condition can be determined by the slope of h at equilibrium point.

Proposition 3. From (2.4), if $h(N_t) > N_t$, the network size will increase, otherwise decrease.

Proof. If $h(N_t) = 1 - F(p - g(N_t)) > N_t$, implies $F^{-1}(1 - N_t) = v' > p - g(N_t)$. Hence, the customers whose individual preference $v^* < v < v'$ will join network.

Proposition 4. From (2.4), if $h'(\bar{N}) < 1$, \bar{N} is a stable equilibrium, otherwise unstable.

Proof. Let ε be a small number, then $[\bar{N} - \varepsilon, \bar{N} + \varepsilon]$ is a closed ball neighborhood of \bar{N} . If $h'(\bar{N}) < 1$, $\lim_{\varepsilon \rightarrow 0} \frac{h(\bar{N} + \varepsilon) - h(\bar{N})}{\varepsilon} < 1$ implies $\lim_{\varepsilon \rightarrow 0} \frac{h(\bar{N} + \varepsilon) - (\bar{N} + \varepsilon) - \{h(\bar{N}) - (\bar{N})\}}{\varepsilon} < 1$. Because \bar{N} is the equilibrium point of h , $\lim_{\varepsilon \rightarrow 0} \frac{h(\bar{N} + \varepsilon) - (\bar{N} + \varepsilon)}{\varepsilon} < 0$. From proposition 3, for the right-hand neighborhood

($\varepsilon > 0$), implies $h(\bar{N} + \varepsilon) < (\bar{N} + \varepsilon)$, and the network size will decrease. And for left-hand neighborhood ($\varepsilon < 0$), the network size will increase. In other words, if $h'(\bar{N}) < 1$, the neighborhood of equilibrium will move toward equilibrium. Hence, it is stable.

The stability condition of SE scenario as well as RE scenario depends on the first derivative of h . Since the equilibrium condition of SE scenario is the first order difference equation in which the characteristic equation linearized around \bar{N} is $\lambda - h'(\bar{N}) = 0$, and the equilibrium points are considered stable if $|\lambda| < 1$, otherwise unstable. The stability condition of SE scenario is the same as RE scenario, that is

$$h'(\bar{N}) < 1 \tag{4.1}$$

Moreover, the solution path N_t will oscillate if $\lambda < 0$. But h is an increasing function; the solution path N_t is oscillation-free, see Figure 4.1(a). In area (1), the equilibrium points are stable and their solution paths are oscillation-free; while in area (2), the equilibrium points are unstable.

For CAE scenario, the equilibrium condition (3.2) is also the first difference equation in which the characteristic equation linearized around \bar{N} is $\lambda - (1 - \omega) - \omega h'(\bar{N}) = 0$, and the equilibrium points are considered to be stable if $|\lambda| < 1$, otherwise unstable.

The stability condition of CAE scenario is $|H'| < 1$, that is,

$$h'(\bar{N}) < 1 \text{ and } h'(\bar{N}) > 1 - \frac{2}{\omega} \tag{4.2}$$

and the oscillating condition is $H' < 0$, that is

$$h'(\bar{N}) > 1 - \frac{1}{\omega} \tag{4.3}$$

Obviously, these conditions are dependent on $h'(\bar{N})$ and , see Figure 4.1(b). In area (1), the equilibrium points are stable and their solution paths are oscillation-free. In area (2), the equilibrium points are stable but their solution paths oscillate; while in area (3) and (4), the equilibrium points are unstable.

The last scenario, LAE, the equilibrium condition (3.5) is the second order difference equation which the characteristic equation linearized around \bar{N} is $\lambda^2 - (1 - \omega)\lambda - \omega h'(\bar{N}) = 0$; which the root is $\lambda_1, \lambda_2 = \frac{1-\omega}{2} \pm \sqrt{\left(\frac{1-\omega}{2}\right)^2 + \omega h'(\bar{N})}$. Also, the equilibrium points are considered stable if both roots lie in the unit circle, otherwise unstable. This condition is the same as $|1 - \omega| < 1 - \omega h'(\bar{N}) < 2$ (Sysdaeter, 1995), that is,

$$h'(\bar{N}) < 1 \text{ and } h'(\bar{N}) < \frac{2}{\omega} - 1 \quad (4.4)$$

Since $h'(\bar{N}) > 0$ and $\omega > 0$, λ_1, λ_2 are not the complex number. Then, the solution path will oscillate if the dominant root is negative, see Simonovits (2000). The oscillating condition of LAE scenario is

$$\omega > 1 \quad (4.5)$$

From Figure 4.1(c), in area (1), the equilibrium points are stable and their solution paths are oscillation-free; in area (2), the equilibrium points are stable but their solution paths oscillate; while in area (3) and (4), the equilibrium points are unstable.

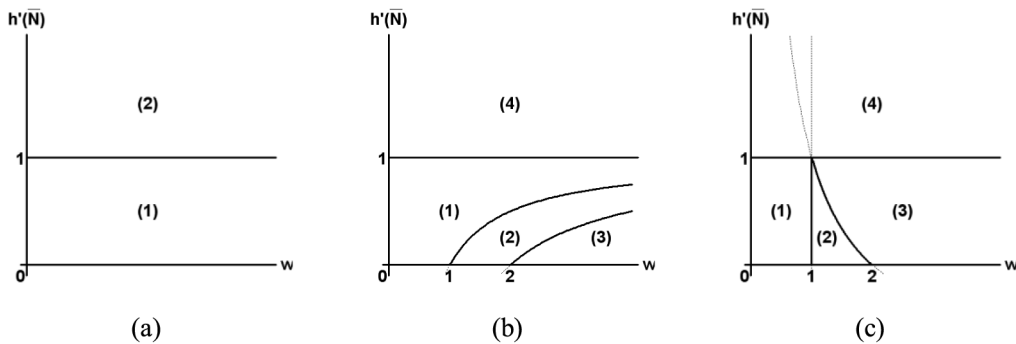


Figure 4.1 The stability and oscillating regions of (a) SE (b) CAE and (c) LAE scenario.

From (4.1)-(4.5), the stability and oscillating conditions depend on $h'(\bar{N})$ and ω . From (2.3) $h'(\bar{N}) > 0$, and the range of speed of adjustment is $0 < \omega \leq 1$, then $h'(\bar{N}) < 1$ is a subset of $h'(\bar{N}) > 1 - \frac{1}{\omega}$, $h'(\bar{N}) > 1 - \frac{2}{\omega}$ and $h'(\bar{N}) < \frac{2}{\omega} - 1$. In other words, the stability and oscillating conditions of all expectation

scenarios are the same. Due to the economic interpretation, the speed of adjustment is set to be $0 < \omega \leq 1$, however, in mathematics, it is possible to extend its range to, $\omega > 0$. In such case, the stability and oscillating conditions of four expectations are different; see Figure 4.2 and Table 4.1.

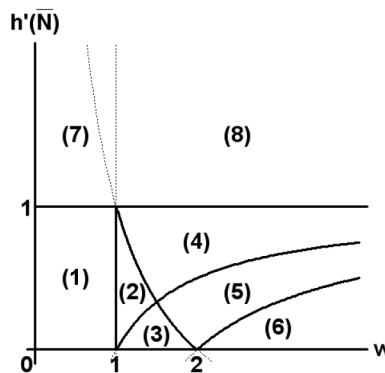


Figure 4.2 The stability and oscillating regions of four expectations

Concerning the economics interpretation $0 < \omega \leq 1$, the stability of equilibrium points and oscillation of solution paths are regardless of expectation type. Regarding to the extension of the range $\omega > 0$, the stability of equilibrium points does not depend on the expectation type in area

(1), (2), (3), (7) and (8) but depends on the expectation type in area (4), (5) and (6); while the oscillation of solution paths does not depend on the expectation type in area (1) and (7) and depends on the expectation type in area (2), (3), (4), (5), (6) and (8).

Table 4.1: The stability of equilibrium point and oscillation of solution path of four expectations

Area	RE+SE		CAE		LAE	
	Stability	Oscillation	Stability	Oscillation	Stability	Oscillation
1	yes	no	yes	no	yes	yes
2	yes	no	yes	no	yes	no
3	yes	no	yes	yes	yes	no
4	yes	no	yes	no	no	no
5	yes	no	yes	yes	no	no
6	yes	no	no	yes	no	no
7	no	no	no	no	no	yes
8	no	no	no	no	no	no

5. Initial Point

In previous section, we discuss about the stability of equilibrium points and found that the actual and expected network size converge to the stable equilibrium point. However, in network goods, the multiple equilib-

riums are commonly assumed, so then the location of equilibrium is determined by the initial expected network size of customer, N_0^e . According to the illustration 1, there are three equilibrium points. Figure 5.1 is the solution paths of SE and CAE scenarios for various initial points.

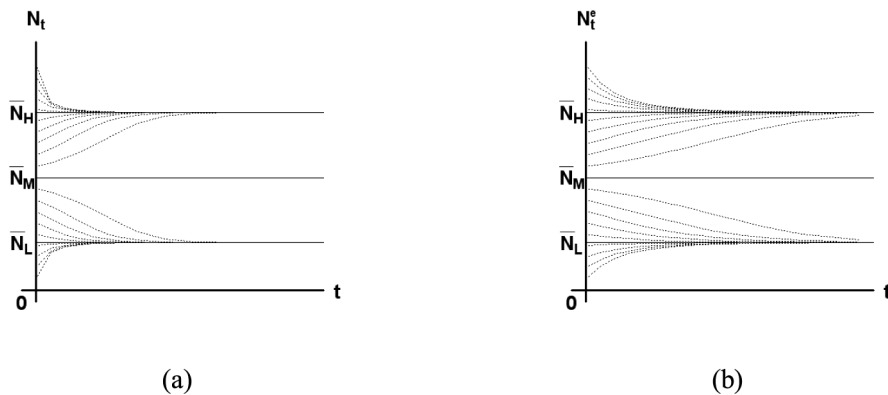


Figure 5.1 The solution path for various initial points of SE (a) and CAE (b)

For SE scenario along with CAE scenario, if the initial point starts in $[0, \bar{N}_M)$, the solution paths will converge to \bar{N}_L ; if the initial point starts at \bar{N}_M , the solution paths will not move away from \bar{N}_M ; and if the initial point starts in $(\bar{N}_M, 1]$, the solution paths will converge to \bar{N}_H . Moreover, the adjustment process of SE scenario is faster than CAE scenario. In case of LAE scenario, we

need a pair of initial points, the first and second period, (N_0^e, N_1^e) . Figure 5.2 is a phase diagram of LAE scenario for various initial points. The solution paths will converge to an either the lower or higher stable equilibrium point, if any pairs of initial points start in area (1) and area (2), respectively.

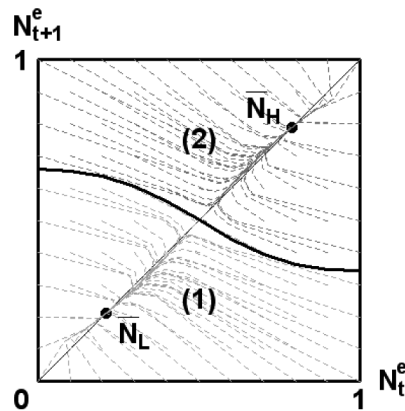


Figure 5.2 The phase diagram for various pairs of initial points of LAE scenario

6. Conclusion

In the study of network goods, the network size has influence on customers' utility. Therefore, customers must speculate the network size through their expectations. In this paper, we review two well-known expectations, rational and static expectations. In addition, we introduce the other well-known expectation, adaptive expectation, with current and lagged information. From literature, the locations of equilibrium points and their stability are free from the expectation type. Our result, however, differently concludes. We have the same conclusion about the locations of equilibrium points, but not their stability. The stability and the oscillation do not only depend on the characteristic of function h but also on the range of speed of adjustment ω . The key result is that if we extend the range of speed of adjustment from

$0 < \omega \leq 1$ to $\omega > 0$, the stability and oscillation will depend on the expectation type. However, our objective is not to discover the non-robustness when expectation type changes. Rather, this paper helps understand the other factors that affect stability of equilibrium points and cautions us to assume the expectation type in network goods.

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